A Concrete Category of Classical Proofs

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Download slides here: http://consequently.org/presentation/2017/a-category-of-classical-proofs-tacl

My Aim

To show how *proof terms* for classical propositional logic form a *category*, and to examine some of its properties.

Proof Terms

The Proof Term Category

It's not *Cartesian*

It is *Monoidal*, and more…

Isomorphisms

Further Work

PROOF TERMS

There can be different ways to prove the same thing

$$p \wedge q \succ p \vee q$$

Four different derivations,

$$\dfrac{\dfrac{p \succ p}{p \land q \succ p}\ \land L}{p \land q \succ p \lor q}\ \lor R \qquad \dfrac{\dfrac{p \succ p}{p \succ p \lor q}\ \lor R}{p \land q \succ p \lor q}\ \land L$$

$$\dfrac{\dfrac{q \succ q}{p \land q \succ q}\ \land L}{p \land q \succ p \lor q}\ \lor R \qquad \dfrac{\dfrac{q \succ q}{q \succ p \lor q}\ \lor R}{p \land q \succ p \lor q}\ \land L$$

Four different derivations, two *proofs*

$$\dfrac{\dfrac{p \succ p}{p \land q \succ p}\,{\land L}}{p \land q \succ p \lor q}\,{\lor R} \qquad \dfrac{\dfrac{p \land q}{p}}{p \lor q} \qquad \dfrac{\dfrac{p \succ p}{p \succ p \lor q}\,{\lor R}}{p \land q \succ p \lor q}\,{\land L}$$

$$\dfrac{\dfrac{q \succ q}{p \land q \succ q}\,{\land L}}{p \land q \succ p \lor q}\,{\lor R} \qquad \dfrac{\dfrac{p \land q}{q}}{p \lor q} \qquad \dfrac{\dfrac{q \succ q}{q \succ p \lor q}\,{\lor R}}{p \land q \succ p \lor q}\,{\land L}$$

Motivating Idea

Proof terms are an *invariant*
for derivations under rule permutation.

δ_1 and δ_2 have the same *term* iff
some permutation sends δ_1 to δ_2.

Four different derivations, two *proof terms*

$$\dfrac{\dfrac{\overset{x \curvearrowright y}{x : p \succ y : p}}{\overset{\acute{\wedge}x \curvearrowright \acute{\vee}y}{x : p \wedge q \succ y : p}}\ {\scriptstyle \wedge L}}{\overset{\acute{\wedge}x \curvearrowright \acute{\vee}y}{x : p \wedge q \succ y : p \vee q}}\ {\scriptstyle \vee R}
\qquad \acute{\wedge}x \curvearrowright \acute{\vee}y \qquad
\dfrac{\dfrac{\overset{x \curvearrowright x}{x : p \succ y : p}}{\overset{x \curvearrowright \acute{\vee}y}{x : p \succ y : p \vee q}}\ {\scriptstyle \vee R}}{\overset{\acute{\wedge}x \curvearrowright \acute{\vee}y}{x : p \wedge q \succ y : p \vee q}}\ {\scriptstyle \wedge L}$$

$$\dfrac{\dfrac{\overset{x \curvearrowright y}{x : q \succ y : q}}{\overset{\grave{\wedge}x \curvearrowright y}{x : p \wedge q \succ y : q}}\ {\scriptstyle \wedge L}}{\overset{\grave{\wedge}x \curvearrowright \grave{\vee}y}{x : p \wedge q \succ y : p \vee q}}\ {\scriptstyle \vee R}
\qquad \grave{\wedge}x \curvearrowright \grave{\vee}y \qquad
\dfrac{\dfrac{\overset{x \curvearrowright y}{x : q \succ y : q}}{\overset{x \curvearrowright \grave{\vee}y}{x : q \succ y : p \vee q}}\ {\scriptstyle \vee R}}{\overset{\grave{\wedge}x \curvearrowright \grave{\vee}y}{x : p \wedge q \succ y : p \vee q}}\ {\scriptstyle \wedge L}$$

λ terms

Ingredients

λ terms ♦ flow graphs

Ingredients

λ terms ⧫ flow graphs ⧫ proof nets

A *proof term* for $\Sigma \succ \Delta$
encodes the flow of information
in a proof of $\Sigma \succ \Delta$.

Results

- Cut elimination is *confluent* and *terminating*.

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  [So it can be understood as a kind of *evaluation*.]

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- Cut elimination for proof terms is *local*.

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- Cut elimination is *confluent* and *terminating*.

  [So it can be understood as a kind of *evaluation*.]

- Cut elimination for proof terms is *local*.

  [So it is easily made parallel.]

Proof Terms

See http://consequently.org/writing/

PROOF TERMS FOR CLASSICAL DERIVATIONS

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Abstract: I give an account of *proof terms* for derivations in a sequent calculus for classical propositional logic. The term for a derivation δ of a sequent $\Sigma \succ \Delta$ encodes *how* the premises Σ and conclusions Δ are related in δ. This encoding is many–to–one in the sense that different derivations can have the same proof term, since different derivations may be different ways of representing the same underlying connection between premises and conclusions. However, not all proof terms for a sequent $\Sigma \succ \Delta$ are the same. There may be *different* ways to connect those premises and conclusions.

Proof terms can be simplified in a process corresponding to the elimination of cut inferences in sequent derivations. However, unlike cut elimination in the sequent calculus, each proof term has a *unique normal form* (from which all cuts have been eliminated) and it is straightforward to show that term reduction is strongly normalising—*every* reduction process terminates in that unique normal form. Furthermore, proof terms are *invariants* for sequent derivations in a strong sense—two derivations δ_1 and δ_2 have the same proof term *if and only if* some permutation of derivation steps sends δ_1 to δ_2 (given a relatively natural class of permutations of derivations in the sequent calculus). Since not every derivation of a sequent can be permuted into every other derivation of that sequent, proof terms provide a non-trivial account of the identity of proofs, independent of the syntactic representation of those proofs.

OUTLINE

Proof Terms

$$\acute{\wedge}x \curvearrowright \acute{\wedge}\,\acute{\vee}y \quad \acute{\wedge}x \curvearrowright \acute{\wedge}\,\grave{\vee}y \quad \acute{\vee}\,\grave{\wedge}x \curvearrowright \grave{\wedge}\,\acute{\vee}y \quad \grave{\vee}\,\grave{\wedge}x \curvearrowright \grave{\wedge}\,\grave{\vee}y$$

$$x : p \wedge (q \vee r) \succ y : (p \wedge q) \vee (p \wedge r)$$

$$\acute{\wedge}x \curvearrowright \acute{\wedge}\acute{\vee}y \quad \acute{\wedge}x \curvearrowright \acute{\wedge}\grave{\vee}y \quad \acute{\vee}\grave{\wedge}x \curvearrowright \grave{\wedge}\acute{\vee}y \quad \grave{\vee}\grave{\wedge}x \curvearrowright \grave{\wedge}\grave{\vee}y$$

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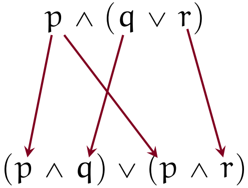
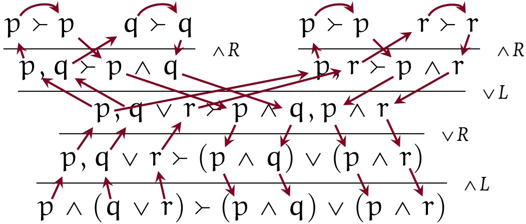


Finding a Proof Term from a Derivation



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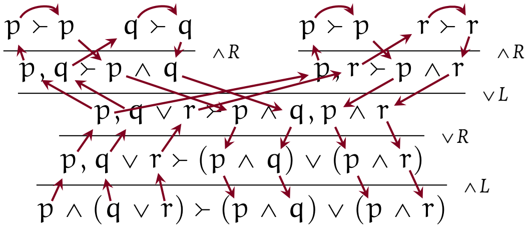

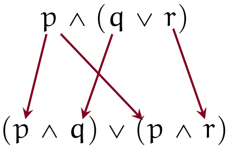

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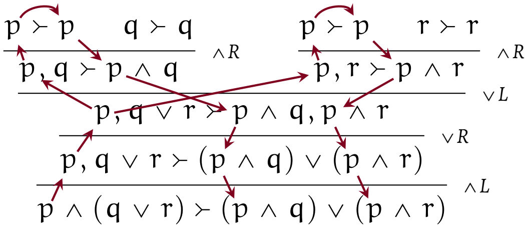
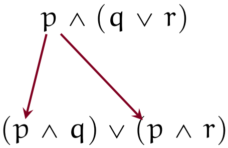


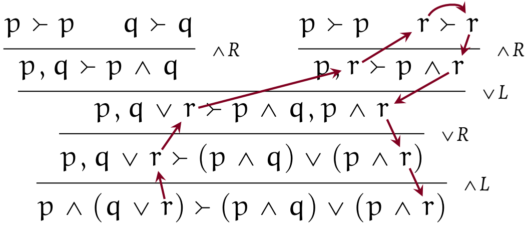


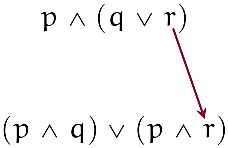
$$
\dfrac{\dfrac{\dfrac{\dfrac{p \succ p \qquad q \succ q}{p, q \succ p \wedge q}\,{\scriptstyle \wedge R} \qquad \dfrac{p \succ p \qquad r \succ r}{p, r \succ p \wedge r}\,{\scriptstyle \wedge R}}{p, q \vee r \succ p \wedge q, p \wedge r}\,{\scriptstyle \vee L}}{p, q \vee r \succ (p \wedge q) \vee (p \wedge r)}\,{\scriptstyle \vee R}}{p \wedge (q \vee r) \succ (p \wedge q) \vee (p \wedge r)}\,{\scriptstyle \wedge L}
$$

$$p \wedge (q \vee r)$$

$$(p \wedge q) \vee (p \wedge r)$$

Finding a Proof Term from a Derivation





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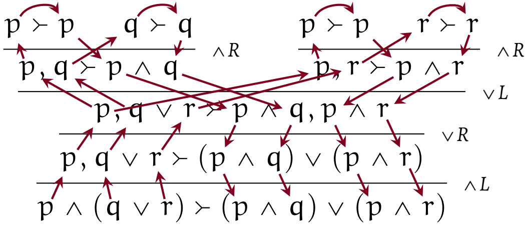
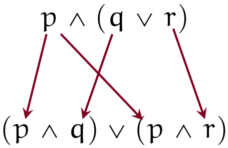
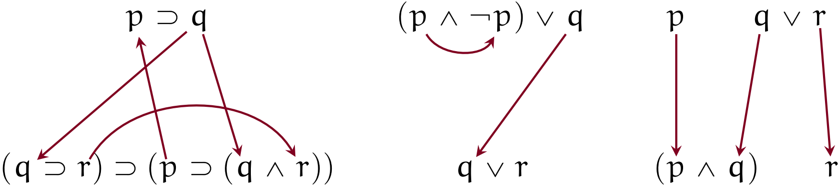




More Flow Graphs



Proof Term Facts

Not every directed graph on occurrences of atoms in a sequent is a proof term.

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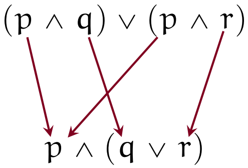
Proof Term Facts

Not every directed graph on occurrences of atoms in a sequent is a proof term.

- They *typecheck*. [An occurrence of p is linked only with an occurrence of p.]
- They *respect polarities*. [Positive occurrences of atoms in premises and negative occurrences of atoms in conclusions are *inputs*. Negative occurrences of atoms in premises and positive occurrences of atoms in conclusions are *outputs*.]
- They must satisfy an "enough connections" condition, amounting to a non-emptiness under every *switching*. [e.g. the obvious linking between premise $p \vee q$ and conclusion $p \wedge q$ is not connected enough to be a proof term.]

Cut is chaining of proof terms



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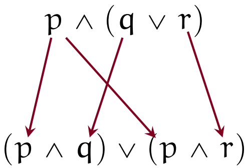

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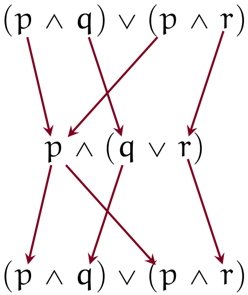
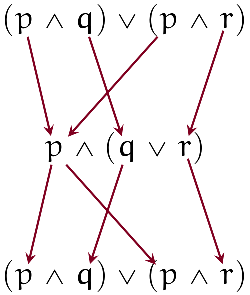
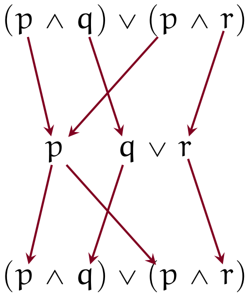
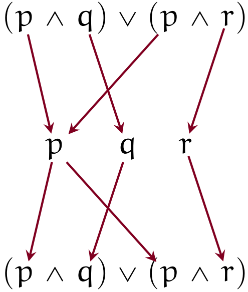


The *cut formula* is no longer a premise or a conclusion in the proof term.

Eliminating Cuts is Local




(p ∧ q) ∨ (p ∧ r)
p
q ∨ r
(p ∧ q) ∨ (p ∧ r)


(p ∧ q) ∨ (p ∧ r)
p
q
r
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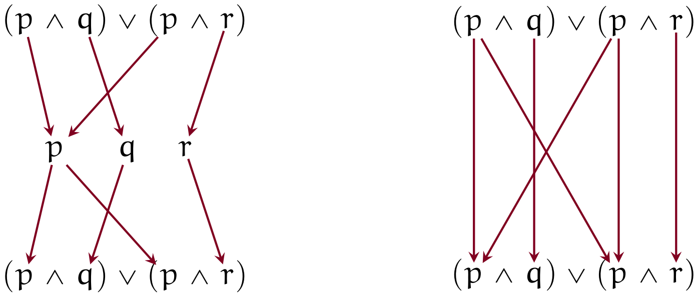


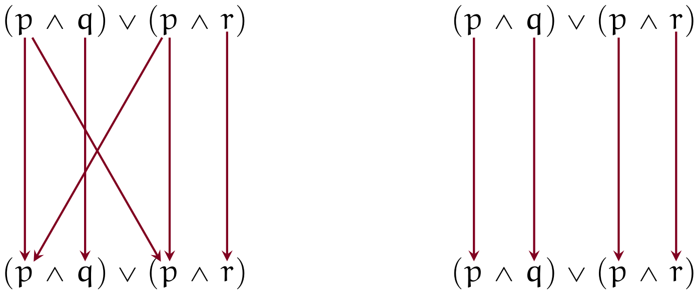
The Conjunction Reduction Case, for Derivations

$$
\dfrac{
\dfrac{
\begin{array}{c}\delta_1 \\ \Sigma_1 \succ A, \Delta_1\end{array}
\qquad
\begin{array}{c}\delta_2 \\ \Sigma_2 \succ B, \Delta_1\end{array}
}{\Sigma_{1,2} \succ A \wedge B, \Delta_{1,2}}\ {\wedge R}
\qquad
\dfrac{
\begin{array}{c}\delta_3 \\ \Sigma_3, A, B \succ \Delta_3\end{array}
}{\Sigma_3, A \wedge B \succ \Delta_3}\ {\wedge L}
}{\Sigma_{1-3} \succ \Delta_{1-3}}\ \mathit{Cut}_{A \wedge B}
$$

reduces to

$$
\dfrac{
\begin{array}{c}\delta_1 \\ \Sigma_1 \succ A, \Delta_1\end{array}
\qquad
\dfrac{
\begin{array}{c}\delta_2 \\ \Sigma_2 \succ B, \Delta_1\end{array}
\qquad
\begin{array}{c}\delta_3 \\ \Sigma_3, A, B \succ \Delta_3\end{array}
}{\Sigma_{2,3}, A \succ \Delta_{2,3}}\ \mathit{Cut}_B
}{\Sigma_{1-3} \succ \Delta_{1-3}}\ \mathit{Cut}_A
$$

Two Different Proofs from $(p \land q) \lor (p \land r)$ to itself



The *second* proof term is the *identity* proof.

Bounds

$\top$ and $\bot$ are interesting.

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They act like $p \vee \neg p$ and $q \wedge \neg q$,
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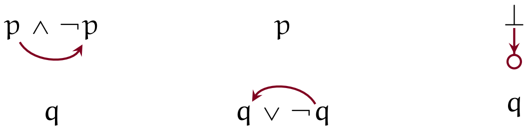

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They act like $p \vee \neg p$ and $q \wedge \neg q$,
except they have no internal structure.

A $\bot$ link has an input but no output.

A $\top$ link has an output but no input.

No links have $\top$ as an input.

No links have $\bot$ as an output.

$$\begin{array}{c} A \\ \Downarrow \\ A \end{array}$$

This is defined in the obvious way.

$$\begin{array}{c} A \\ \Big\Downarrow \\ A \end{array}$$

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$$\begin{array}{c} p \\ \Big\downarrow \\ p \end{array}$$

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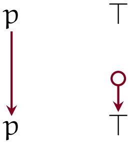

Identity Proofs

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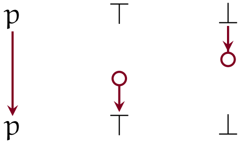

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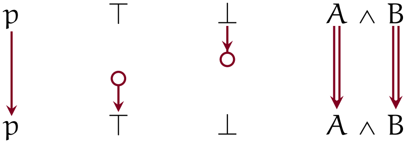

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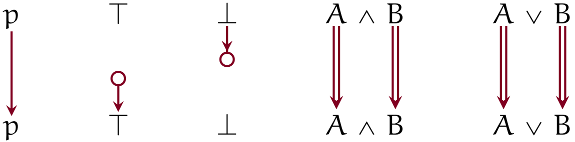

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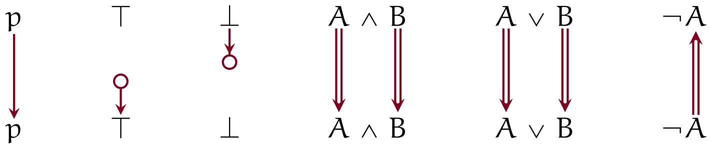

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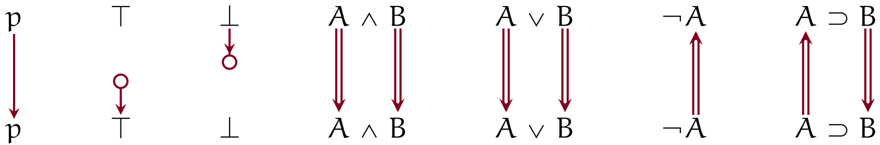


THE PROOF TERM
CATEGORY

OBJECTS: *Formulas* — ARROWS: *Cut-Free Proof Terms*

- $\pi : A \to B$ iff $\pi(x)[y]$ is a *cut-free* proof for $x : A \succ y : B$.

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- $id_A : A \longrightarrow A$ is the identity proof term $x \curvearrowright y$ of type A.
- Composition is chaining proofs & elimination of cuts.
  - If $\pi : A \to B$ and $\tau : B \to C$ then $\tau \circ \pi : A \to C$ is $(\pi(x)[\bullet]\ \tau(\bullet)[y])^*$.

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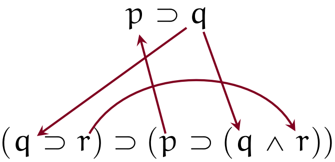
- Composition is chaining proofs & elimination of cuts.
  - If $\pi : A \to B$ and $\tau : B \to C$ then $\tau \circ \pi : A \to C$ is $(\pi(x)[\bullet]\ \tau(\bullet)[y])^*$.
- Composition is associative.
- Identity proofs are indeed identities in the category:
  - $(\pi(x)[\bullet]\ \bullet \frown y)^* = \pi(x)[y]$, and $(x \frown \bullet\ \ \pi(\bullet)[y])^* = \pi(x)[y]$, when π is cut-free.

How Identity Proofs Compose



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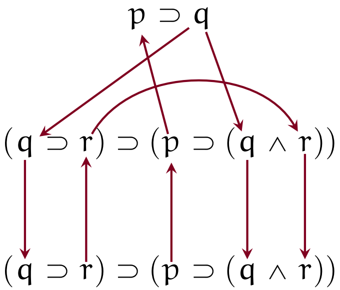

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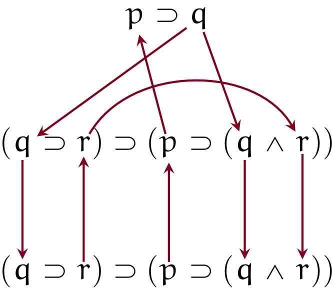
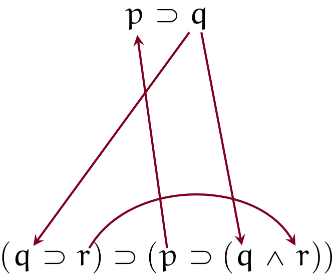




We have a Category

Proof Terms

- π has type $\Sigma \succ \Delta$.
- Proofs are SET–SET.
- Proofs include *Cuts*.

$$\dfrac{\overset{x\curvearrowright y}{x : A \succ y : A} \qquad \overset{\pi(x)[y]}{x : A \succ y : B}}{\overset{x\curvearrowright \bullet \quad \pi(\bullet)[y]}{x : A \succ y : B}}\ Cut$$

The Category $\mathcal{T}$ of Cut-Free Terms

- $\pi : A \rightarrow B$.
- Proofs are FMLA–FMLA.
- Proofs have no *Cuts*.

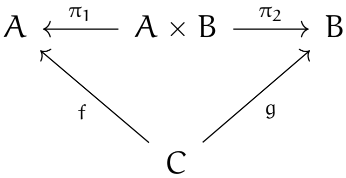
$$\dfrac{A \xrightarrow{id_A} A \qquad A \xrightarrow{\pi} B}{A \xrightarrow{\pi \circ id_A \,=\, \pi} B}$$

What is the proof term category like?

Cartesian Products

$$A \xleftarrow{\pi_1} A \times B \xrightarrow{\pi_2} B$$

Cartesian Products



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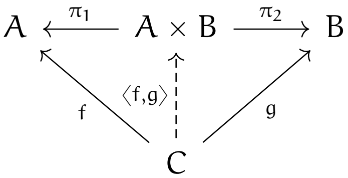

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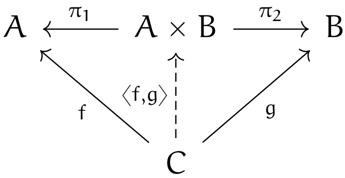
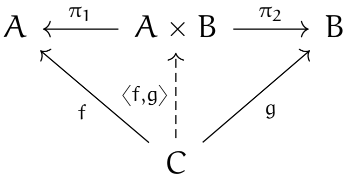


$$\pi_1 \circ \langle f, g \rangle = f \quad \pi_2 \circ \langle f, g \rangle = g$$

Cartesian Products



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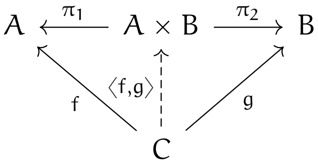
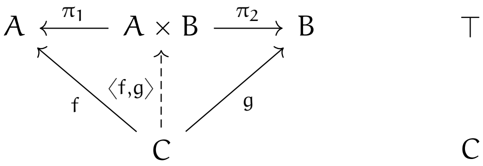
This looks a lot like conjunction.

Many interesting categories have cartesian products.

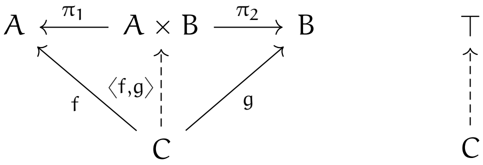
The Empty Product



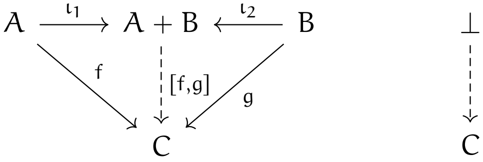
The Empty Product



The Empty Product

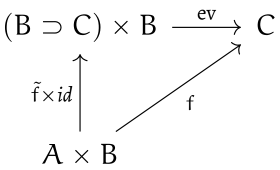


Coproducts and Initial Objects



Residuating Products — internalising arrows

$f : A \times B \to C \qquad \tilde{f} : A \to B \supset C \qquad ev : (B \supset C) \times B \to C$



Cartesian Closed Categories...

…model intuitionistic logic.

…model intuitionistic logic.

They collapse into preorders when made classical.

So what is the proof term category?

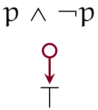
So what is the proof term category?

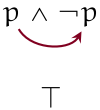
Since it isn't a preorder, and it is classical...


IT’S NOT CARTESIAN

$\top$ is not Terminal, $\bot$ is not Initial

⊤ is not Terminal, ⊥ is not Initial





⊤ is not Terminal, ⊥ is not Initial

… and nothing else is initial or terminal either

If T is a candidate terminal object,
then there is some arrow $\top \to \mathsf{T}$.

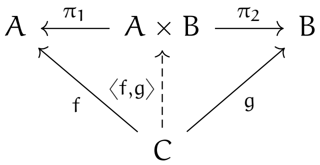
In this arrow all links are internal to T
(since $\top$ is never a source in a link).

These links generate a proof term for $\bot \to \mathsf{T}$,
and this proof ignores $\bot$.

There is a different proof term for $\bot \to \mathsf{T}$
using $\bot$ and ignoring T.

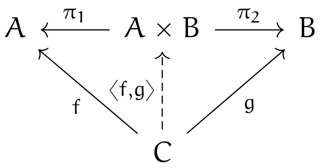
(This dualises for any candidate initial object I.)

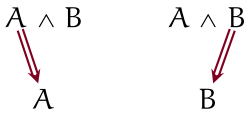
Conjunction isn't Cartesian Product



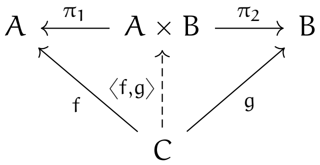
Conjunction isn't Cartesian Product

We have candidate projection arrows.



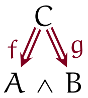


Conjunction isn't Cartesian Product

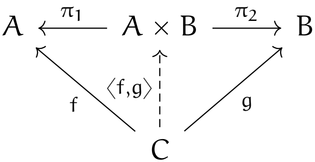


We have candidate projection arrows.

And a candidate pairing arrow.



Conjunction isn't Cartesian Product



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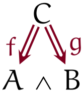
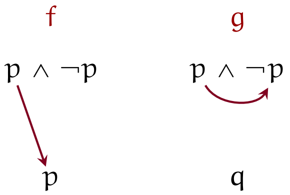
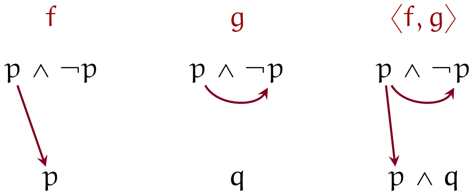


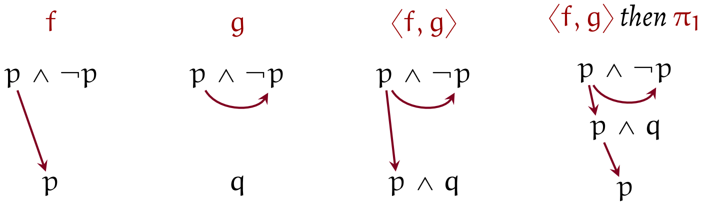
But its composition with "projection" need not restore f and g.

An Example



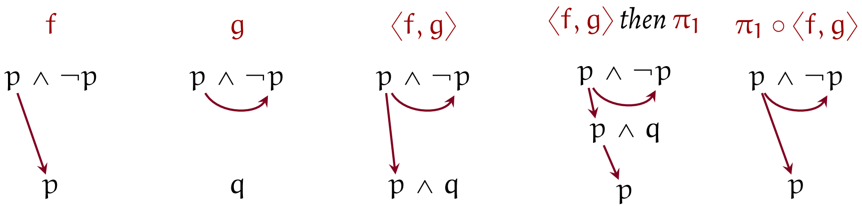

f
g
⟨f, g⟩
p ∧ ¬p
p ∧ ¬p
p ∧ ¬p
p
q
p ∧ q

An Example



f
g
⟨f, g⟩
⟨f, g⟩ *then* π_1
$p \wedge \neg p$
$p \wedge \neg p$
$p \wedge \neg p$
$p \wedge \neg p$
$p \wedge q$
p
q
$p \wedge q$
p

An Example



An Example

Notice: $\pi_1 \circ \langle f, g \rangle$ is not f.

It has some of g (in this case, *all* of the links of g) left behind.

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It has some of g (in this case, *all* of the links of g) left behind.

However, in general, $f \subseteq \pi_1 \circ \langle f, g \rangle$ and $g \subseteq \pi_2 \circ \langle f, g \rangle$.

Diagnosis

This arises from the *locality* of cut reduction.

$$\dfrac{\dfrac{\dfrac{p, \neg p \succ p}{p \wedge \neg p \succ p} \qquad \dfrac{\dfrac{p \succ p, q}{p, \neg p \succ q}}{p \wedge \neg p \succ q}}{p \wedge \neg p \succ p \wedge q}{\scriptstyle \wedge R} \qquad \dfrac{p, q \succ p}{p \wedge q \succ p}{\scriptstyle \wedge L}}{p \wedge \neg p \succ p}{\scriptstyle Cut_{p \wedge q}}$$

$$\rightsquigarrow$$

$$\dfrac{\dfrac{p, \neg p \succ p}{p \wedge \neg p \succ p} \qquad \dfrac{\dfrac{\dfrac{p \succ p, q}{p, \neg p \succ q}}{p \wedge \neg p \succ q} \qquad p, q \succ p}{p, p \wedge \neg p \succ p}{\scriptstyle Cut_{q}}}{p \wedge \neg p \succ p}{\scriptstyle Cut_{p}}$$

In fact, there are *no* Cartesian Products

A *slightly* more general argument shows that there is *no* object $p \times q$

- equipped with projection arrows $\pi_1 : p \times q \to p$ and $\pi_2 : p \times q \to q$,
- where there is some proof $h : p \wedge \neg p \to p \times q$, such that
- $\pi_1 \circ h = f$ and $\pi_2 \circ h = g$.

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- equipped with projection arrows $\pi_1 : p \times q \to p$ and $\pi_2 : p \times q \to q$,
- where there is some proof $h : p \wedge \neg p \to p \times q$, such that
- $\pi_1 \circ h = f$ and $\pi_2 \circ h = g$.

(The argument is a dilemma: does h contain a link between the instances of p in the premise $p \wedge \neg p$? If it *does*, then composition with π_1 preserves that link, and $\pi_1 \circ h$ isn't f. If it *doesn't*, there is no way for $\pi_2 \circ h$ to contain that link.)

So, if it isn't Cartesian, what *is* the category like?


IT IS MONOIDAL,
& MORE…

Many categories have something *like* cartesian product, but different.

Monoidal Categories

Many categories have something *like* cartesian product, but different.

Tensor product — $\otimes$ — in vector spaces is an important example.

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Tensor product — $\otimes$ — in vector spaces is an important example.

This motivates the definition of a *monoidal* category.

Symmetric Monoidal Categories

$$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \qquad 1 \in Ob(\mathcal{C})$$

Symmetric Monoidal Categories

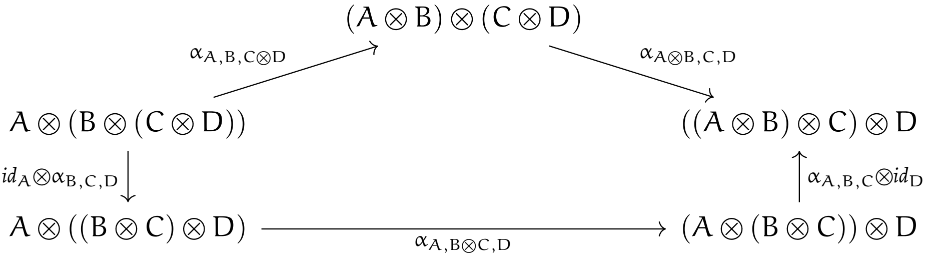
$$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \qquad 1 \in Ob(\mathcal{C})$$

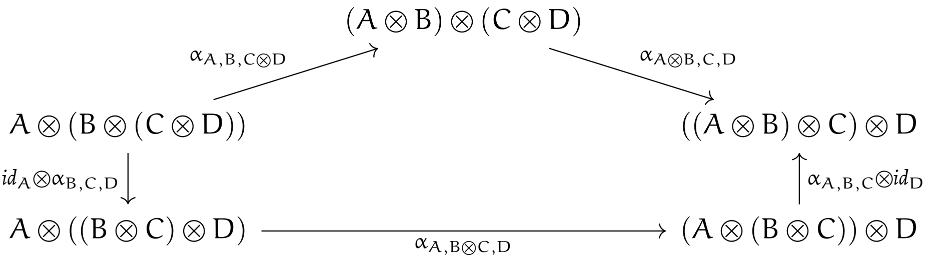
$$\alpha_{A,B,C} : A \otimes (B \otimes C) \xrightarrow{\sim} (A \otimes B) \otimes C$$

$$\sigma_{A,B} : A \otimes B \xrightarrow{\sim} B \otimes A \qquad \iota_A : 1 \otimes A \xrightarrow{\sim} A$$

where *associativity* (α), *symmetry* (σ) and *unit* (ι) behave sensibly.

Associativity





(The ‘Pentagon’)

Symmetry

$$A \otimes B \underset{\sigma_{B,A}}{\overset{\sigma_{A,B}}{\rightleftarrows}} B \otimes A$$

Symmetry

$$A \otimes B \underset{\sigma_{B,A}}{\overset{\sigma_{A,B}}{\rightleftarrows}} B \otimes A$$

$$\begin{array}{ccccc}
(A \otimes B) \otimes C & \xrightarrow{\alpha_{A,B,C}} & A \otimes (B \otimes C) & \xrightarrow{\sigma_{A,B\otimes C}} & (B \otimes C) \otimes A \\
\sigma_{A,B} \otimes id_C \downarrow & & & & \downarrow \alpha_{B,C,A} \\
(B \otimes A) \otimes C & \xrightarrow[\alpha_{B,A,C}]{} & B \otimes (A \otimes C) & \xrightarrow[id_B \otimes \sigma_{A,C}]{} & B \otimes (C \otimes A)
\end{array}$$

(The ‘Hexagon’)

(Let's drop the subscripts on α, σ, ι, *id* where there's no ambiguity.)

Unit

$$
\begin{array}{ccc}
(A \otimes 1) \otimes B & \xrightarrow{\alpha} & A \otimes (1 \otimes B) \\
{\scriptstyle \sigma \otimes id} \Big\downarrow & & \Big\downarrow {\scriptstyle id \otimes \iota} \\
(1 \otimes A) \otimes B & \xrightarrow[\iota \otimes id]{} & A \otimes B
\end{array}
$$

(The ‘Square’)

Proof Terms are a Symmetric Monoidal Category under $\wedge/\top$

$$\wedge : \mathcal{T} \times \mathcal{T} \to \mathcal{T} \qquad \top \in Ob(\mathcal{T})$$

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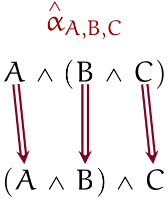
$$\wedge : \mathcal{T} \times \mathcal{T} \to \mathcal{T} \qquad \top \in Ob(\mathcal{T})$$

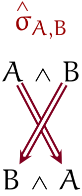
$$\hat{\alpha} : A \wedge (B \wedge C) \xrightarrow{\sim} (A \wedge B) \wedge C$$

$$\hat{\sigma} : A \wedge B \xrightarrow{\sim} B \wedge A \qquad \hat{\iota} : \top \wedge A \xrightarrow{\sim} A$$

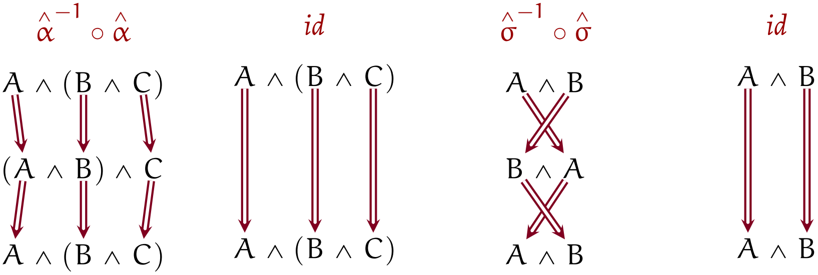
and indeed, *associativity* ($\hat{\alpha}$), *symmetry* ($\hat{\sigma}$) and *unit* ($\hat{\iota}$) behave sensibly.

$\hat{\alpha}$, $\hat{\sigma}$ and $\hat{\iota}$

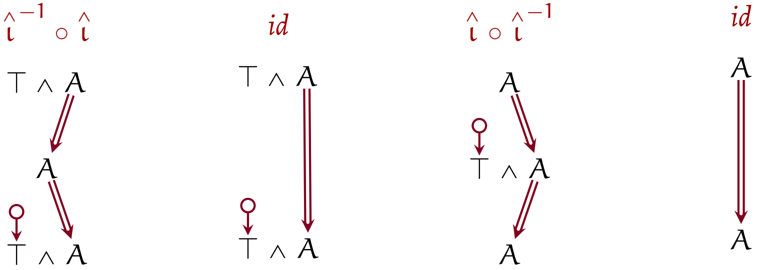




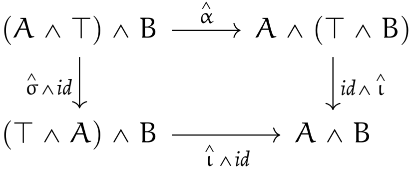
$\hat{\alpha}$, $\hat{\sigma}$ and $\hat{\iota}$ are isomorphisms

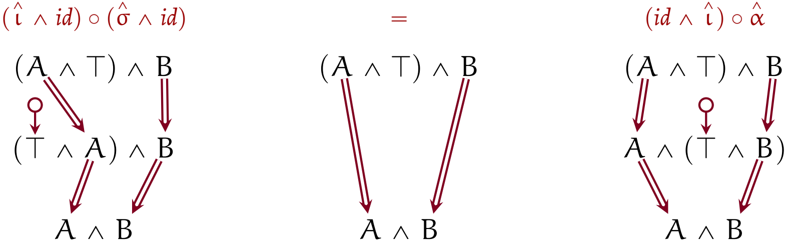


$\hat{\alpha}$, $\hat{\sigma}$ and $\hat{\iota}$ are isomorphisms



The *Pentagon*, *Hexagon*, *Square*, etc., commute





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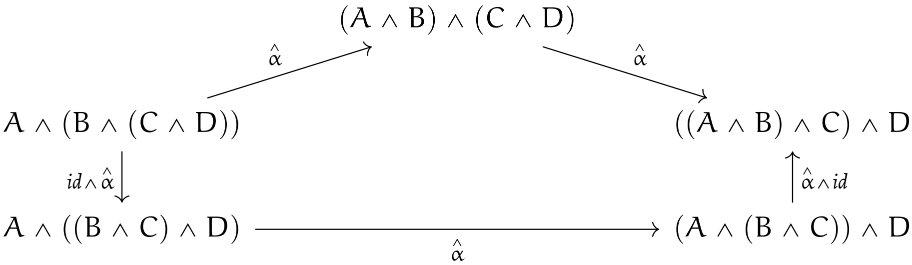

The *Pentagon*, *Hexagon*, *Square*, etc., commute

$$\begin{array}{ccccc}
(A \wedge B) \wedge C & \xrightarrow{\hat{\alpha}} & A \wedge (B \wedge C) & \xrightarrow{\hat{\sigma}} & (B \wedge C) \wedge A \\
{\scriptstyle \hat{\sigma} \wedge id}\downarrow & & & & \downarrow{\scriptstyle \hat{\alpha}} \\
(B \wedge A) \wedge C & \xrightarrow[\hat{\alpha}]{} & B \wedge (A \wedge C) & \xrightarrow[id \wedge \hat{\sigma}]{} & B \wedge (C \wedge A)
\end{array}$$

Proof Terms are a Symmetric Monoidal Category under $\vee/\bot$

$$\vee : \mathcal{T} \times \mathcal{T} \to \mathcal{T} \qquad \bot \in Ob(\mathcal{T})$$

$$\check{\alpha} : A \vee (B \vee C) \xrightarrow{\sim} (A \vee B) \vee C$$

$$\check{\sigma} : A \vee B \xrightarrow{\sim} B \vee A \qquad \check{\iota} : \bot \vee A \xrightarrow{\sim} A$$

and *associativity* ($\check{\alpha}$), *symmetry* ($\check{\sigma}$) and *unit* ($\check{\iota}$) behave *just as* sensibly.

Linear Distributive Categories

The operators $\wedge$ and $\vee$ are connected by δ and δ'

$$\delta : A \wedge (B \vee C) \to (A \wedge B) \vee C \qquad \delta' : (A \vee B) \wedge C \to A \vee (B \wedge C)$$

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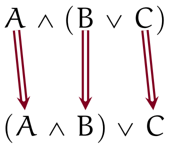
$$\delta : A \wedge (B \vee C) \to (A \wedge B) \vee C \qquad \delta' : (A \vee B) \wedge C \to A \vee (B \wedge C)$$

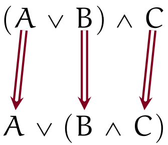
If the operators are *symmetric*, then we need only one.

$$\begin{array}{ccc}
(A \vee B) \wedge C & \xrightarrow{\delta'} & A \vee (B \wedge C) \\
\overset{\wedge}{\sigma} \downarrow & & \uparrow \overset{\vee}{\sigma} \\
C \wedge (A \vee B) & & (B \wedge C) \vee A \\
id \wedge \overset{\vee}{\sigma} \downarrow & & \uparrow \overset{\wedge}{\sigma} \vee id \\
C \wedge (B \vee A) & \xrightarrow[\delta]{} & (C \wedge B) \vee A
\end{array}$$

δ



δ'



Linear Distributivity Conditions

$$
\begin{array}{ccc}
\top \wedge (A \vee B) & & \\
\downarrow{\scriptstyle \delta} & \searrow^{\hat{\iota}} & \\
(\top \wedge A) \vee B & \xrightarrow[\hat{\iota} \vee id]{} & A \vee B
\end{array}
$$

$$
\begin{array}{ccc}
(A \wedge B) \wedge (C \vee D) & \xrightarrow{\hat{\alpha}} & A \wedge (B \wedge (C \vee D)) \\
 & & \downarrow{\scriptstyle id \wedge \delta} \\
\downarrow{\scriptstyle \delta} & & A \wedge ((B \wedge C) \vee D) \\
 & & \downarrow{\scriptstyle \delta} \\
((A \wedge B) \wedge C) \vee D & \xrightarrow[\hat{\alpha} \vee id]{} & (A \wedge (B \wedge C)) \vee D
\end{array}
$$

$$
\begin{array}{ccccc}
((A \vee B) \wedge C) \vee D & \xleftarrow{\delta} & (A \vee B) \wedge (C \vee D) & \xrightarrow{\delta'} & A \vee (B \wedge (C \vee D)) \\
{\scriptstyle \delta' \vee id}\downarrow & & & & \downarrow{\scriptstyle id \vee \delta} \\
(A \vee (B \wedge C)) \vee D & & \xrightarrow[\alpha]{\qquad\qquad\qquad} & & A \vee ((B \wedge C) \vee D)
\end{array}
$$

$$
\begin{array}{ccc}
\top \land (A \lor B) & & \\
\downarrow{\scriptstyle \delta} & \searrow{\scriptstyle \hat\iota} & \\
(\top \land A) \lor B & \xrightarrow[\hat\iota \lor id]{} & A \lor B
\end{array}
$$

$$
\hat\iota : \top \land (A \lor B) \to A \lor B
$$

$$
\begin{array}{ccc}
(A \land B) \land (C \lor D) & \xrightarrow{\hat\alpha} & A \land (B \land (C \lor D)) \\
 & & \downarrow{\scriptstyle id \land \delta} \\
\downarrow{\scriptstyle \delta} & & A \land ((B \land C) \lor D) \\
 & & \downarrow{\scriptstyle \delta} \\
((A \land B) \land C) \lor D & \xrightarrow[\hat\alpha \lor id]{} & (A \land (B \land C)) \lor D
\end{array}
$$

$$
\begin{array}{ccccc}
((A \lor B) \land C) \lor D & \xleftarrow{\delta} & (A \lor B) \land (C \lor D) & \xrightarrow{\delta'} & A \lor (B \land (C \lor D)) \\
\downarrow{\scriptstyle \delta' \lor id} & & & & \downarrow{\scriptstyle id \lor \delta} \\
(A \lor (B \land C)) \lor D & & \xrightarrow[\alpha]{\qquad\qquad} & & A \lor ((B \land C) \lor D)
\end{array}
$$

(These diagrams *clearly* commute in the proof term category.)

There are a number of ways to define *Star-Autonomous* Categories.

We have a $\neg A$ for each object A, and two sets of arrows.

$$\gamma_A : A \wedge \neg A \to \bot \qquad \tau_A : \top \to \neg A \vee A$$

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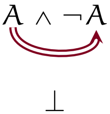
We have a $\neg A$ for each object A, and two sets of arrows.

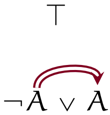
$$\gamma_A : A \wedge \neg A \to \bot \qquad \tau_A : \top \to \neg A \vee A$$

These arrows have natural proof terms.

γ



τ



These Diagrams Must Commute

$$
\begin{array}{ccccc}
A \wedge (\neg A \vee A) & \xrightarrow{\delta} & (A \wedge \neg A) \vee A & \xrightarrow{\gamma \vee id} & \bot \vee A \\
{\scriptstyle id \wedge \tau}\uparrow & & & & \downarrow{\scriptstyle \check{\iota}} \\
A \wedge \top & & \xrightarrow[\hat{\iota}]{\qquad\qquad\qquad} & & A
\end{array}
$$

$$
\begin{array}{ccccc}
(\neg A \vee A) \wedge \neg A & \xrightarrow{\delta'} & \neg A \vee (A \wedge \neg A) & \xrightarrow{id \vee \gamma} & \neg A \vee \bot \\
{\scriptstyle \tau \wedge id}\uparrow & & & & \downarrow{\scriptstyle \check{\iota}} \\
\top \wedge \neg A & & \xrightarrow[\hat{\iota}]{\qquad\qquad\qquad} & & \neg A
\end{array}
$$

These Diagrams Must Commute

$$\begin{array}{ccccc}
A \land (\neg A \lor A) & \xrightarrow{\delta} & (A \land \neg A) \lor A & \xrightarrow{\gamma \lor id} & \bot \lor A \\
\Big\uparrow{\scriptstyle id \land \tau} & & & & \Big\downarrow{\scriptstyle \check{\iota}} \\
A \land \top & & \xrightarrow[\hat{\iota}]{\hspace{6em}} & & A
\end{array}$$

$$\begin{array}{ccccc}
(\neg A \lor A) \land \neg A & \xrightarrow{\delta'} & \neg A \lor (A \land \neg A) & \xrightarrow{id \lor \gamma} & \neg A \lor \bot \\
\Big\uparrow{\scriptstyle \tau \land id} & & & & \Big\downarrow{\scriptstyle \check{\iota}} \\
\top \land \neg A & & \xrightarrow[\hat{\iota}]{\hspace{6em}} & & \neg A
\end{array}$$

These *aren't* so obviously commutative as proof terms.

The negation diagrams commute in the proof term category

$$\begin{array}{ccccc}
A \wedge (\neg A \vee A) & \xrightarrow{\ \delta\ } & (A \wedge \neg A) \vee A & \xrightarrow{\gamma \vee id} & \bot \vee A \\
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A \wedge \top & & \xrightarrow[\hat{\iota}]{\qquad\qquad\qquad} & & A
\end{array}$$

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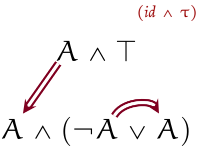
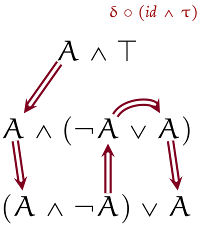
A \wedge \top & & \xrightarrow{\quad\quad\hat{\iota}\quad\quad} & & A
\end{array}
$$

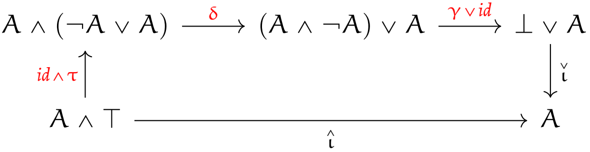


The negation diagrams commute in the proof term category

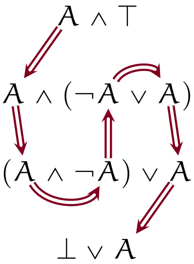
$$\begin{array}{ccccc} A \wedge (\neg A \vee A) & \xrightarrow{\delta} & (A \wedge \neg A) \vee A & \xrightarrow{\gamma \vee id} & \bot \vee A \\ {\scriptstyle id \wedge \tau}\uparrow & & & & \downarrow{\scriptstyle \check{\iota}} \\ A \wedge \top & & \xrightarrow[\hat{\iota}]{\qquad\qquad\qquad} & & A \end{array}$$



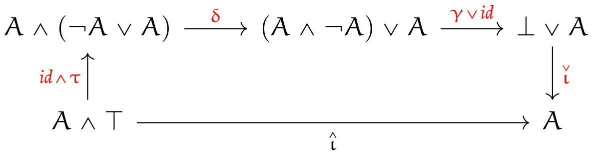
The negation diagrams commute in the proof term category



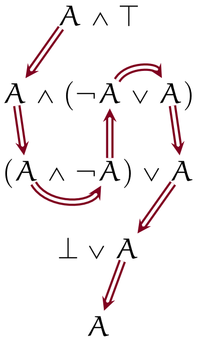
$(\gamma \vee id) \circ \delta \circ (id \wedge \tau)$



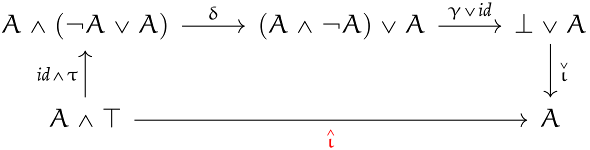
The negation diagrams commute in the proof term category



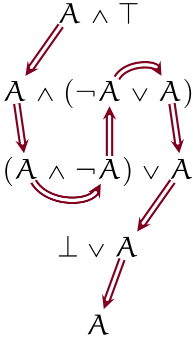
$\check{\iota} \circ (\gamma \vee id) \circ \delta \circ (id \wedge \tau)$



The negation diagrams commute in the proof term category



$\check{\iota} \circ (\gamma \vee id) \circ \delta \circ (id \wedge \tau)$



$\hat{\iota}$

These categories model the multiplicative fragment of linear logic.

Linear Implication

I won't pause now to explain how $A \supset B$, definable as $\neg A \vee B$ (or as $\neg(A \wedge \neg B)$, to which it's isomorphic) is a right adjoint to $\wedge$.

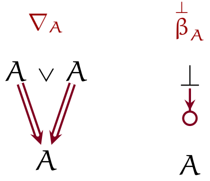
We can do more

Our proof terms allow *contraction* and *weakening*.

Weakening and Contraction Monoids and Comonoids

$$\nabla_A : A \vee A \to A$$

$$\overset{\perp}{\beta}_A : \perp \to A$$



Weakening and Contraction Monoids and Comonoids

$$\nabla_A : A \vee A \to A \qquad \Delta_A : A \to A \wedge A$$

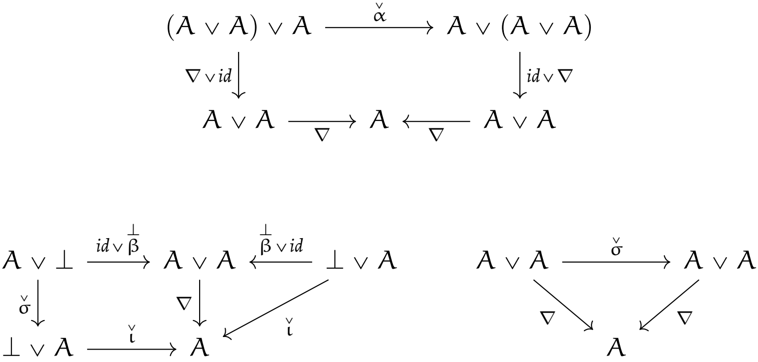
$$\overset{\perp}{\beta}_A : \perp \to A \qquad \overset{\top}{\beta}_A : A \to \top$$

What Makes $\overset{\bot}{\beta}$ and $\overset{\top}{\beta}$ *weakening*?

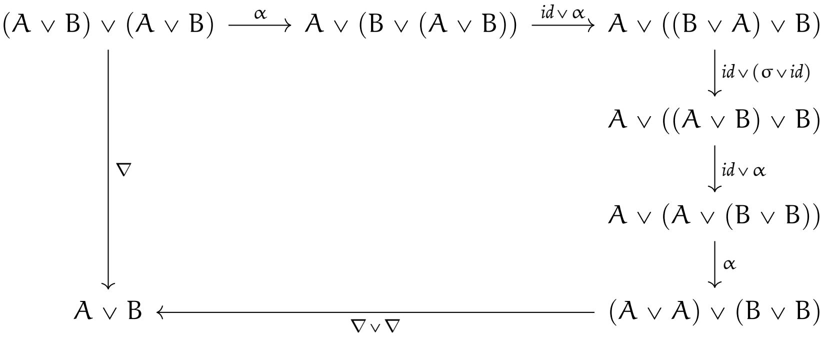
$$A \xrightarrow{\check{\iota}} A \vee \bot \xrightarrow{id \vee \overset{\bot}{\beta}_B} A \vee B$$

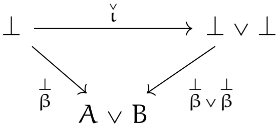
$$A \wedge B \xrightarrow{id \wedge \overset{\top}{\beta}_B} A \wedge \top \xrightarrow{\hat{\iota}} A$$

(Co)monoidal Conditions for Contraction and Weakening



Structurality for ∇ and $\overset{\perp}{\beta}$: disjunctions

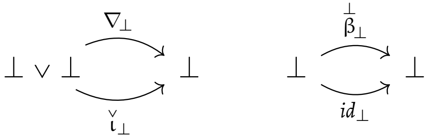




Structurality for ∇ and $\overset{\bot}{\beta}$: bounds

$$\bot \vee \bot \;\overset{\nabla_\bot}{\underset{\check{\iota}_\bot}{\rightrightarrows}}\; \bot \qquad\qquad \bot \;\overset{\overset{\bot}{\beta}_\bot}{\underset{id_\bot}{\rightrightarrows}}\; \bot$$

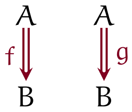
Structurality for ∇ and $\overset{\bot}{\beta}$: bounds

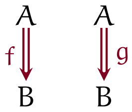


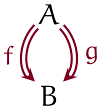
All these conditions are straightforward to verify for proof terms.

And dually for Δ and $\overset{\top}{\beta}$.

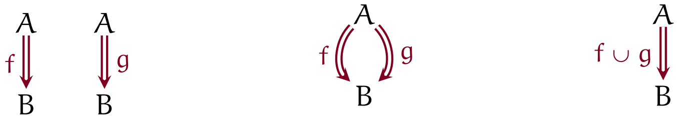
Blend

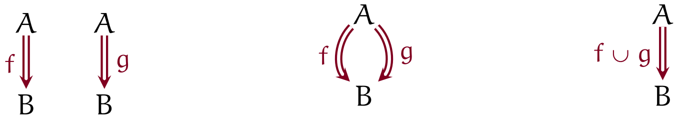



A
f
B
A
g
B


A
f
g
B

Blend





$\cup$ is a semilattice join on $\mathrm{Hom}(A, B)$.

$$(f \cup f') \circ g = (f \circ g) \cup (f' \circ g) \qquad f \circ (g \cup g') = (f \circ g) \cup (f \circ g')$$

The term category $\mathcal{T}$ is *enriched in* SLat.

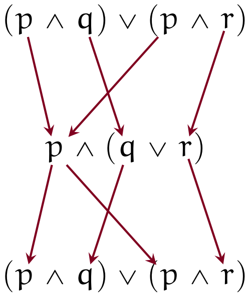
Classical categories are
star autonomous categories
with *structural monoids* and *comonoids*,
enriched in SLat.

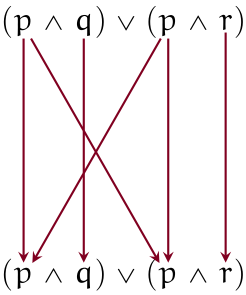
Cf. Führmann and Pym:

- "Order-enriched categorical models of the classical sequent calculus" JPAA (2006)
- "On categorical models of classical logic and the Geometry of Interaction" MSCS (2007).


ISOMORPHISMS

$(p \wedge q) \vee (p \wedge r)$ is not isomorphic to $p \wedge (q \vee r)$





Also Not Isomorphisms

$$p \ncong p \wedge p \qquad p \ncong p \vee p$$

$$p \wedge (p \vee q) \ncong p \vee (p \wedge q) \qquad p \vee \neg p \ncong \top \qquad q \wedge \neg q \ncong \bot$$

Isomorphisms

$$A \wedge \top \cong A \qquad A \wedge B \cong B \wedge A \qquad A \wedge (B \wedge C) \cong (A \wedge B) \wedge C$$

$$A \vee \bot \cong A \qquad A \vee B \cong B \vee A \qquad A \vee (B \vee C) \cong (A \vee B) \vee C$$

$$\neg(A \wedge B) \cong (\neg A \vee \neg B) \qquad \neg(A \vee B) \cong (\neg A \wedge \neg B) \qquad \neg\neg A \cong A$$

$$\neg\top \cong \bot \qquad \neg\bot \cong \top \qquad \top \vee \top \cong \top \qquad \bot \wedge \bot \cong \bot$$

These isomorphisms (together with substitution into arbitrary contexts) *characterise* isomorphism in the term category $\mathcal{T}$.

Inside classical logic,
there is a fine-grained,
hyperintensional notion
of sameness of content,
tighter than logical equivalence
but looser than syntactic identity.


FURTHER WORK

To Do List

- Finish the completeness proof, to the effect that $\mathcal{T}_{\mathcal{L}}$ is the free classical category on $\mathcal{L}$.
- Explore other examples of classical categories.
- Consider the restriction to terms for intuitionist derivations. (This still isn't Cartesian. What sort of category is it?)
- Extend all of this to *first order predicate logic*.

THANK YOU!

http://consequently.org/presentation/2017/
a-category-of-classical-proofs-tacl

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